Lecture 15

Tuesday, 27 October 2026 8:17 PM

Tournament Solutions

or, what to do when there is no Condorcet Winner.

n voters V/N, m alternatives A/C, $\succ_i \in L(A)$, $P = (\succ_1, \dots, \succ_n)$
throughout assume n is odd.
thus majority graph is a tournament.

T (or T_m) is a tournament on m vertices.
Recall: every tournament can be obtained as the majority graph of a profile.
Will think of input as a tournament, rather than a profile.
Notation: $a \to b$ means a beats b

For a directed graph $G = (V, A)$, $u \rightsquigarrow v$ if there is a directed path from u to v. & assume $u \rightsquigarrow u$. Say u, v are strongly connected ($u \sim v$) if $u \rightsquigarrow v$ & $v \rightsquigarrow u$.

Claim: $\sim$ is an equivalence relation
(reflexive, symmetric, transitive)

Defn: $S \subseteq V$ is a strongly connected component if S is an equivalence class under $\sim$.

Consider G^*, obtained by replacing each component in G by a single vertex, and an edge b/w two vertices a, b whenever there is a edge b/w a vtx. in component x & a vtx. in component y.
G^* is called a condensation of G.

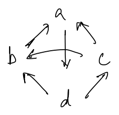
Claim: G^* is acyclic.

Hence G^* is a DAG, & hence the components in G can be written in topological order.

For a tournament, the condensation is an acyclic tournament.

A tournament soln. is a social choice correspondence: $F: T \to 2^A$.
A tournament soln. is Condorcet consistent if, whenever there is a Condorcet winner a, $F(T) = a$

Consider: T_4:



Defn.: $X \subseteq A$ is dominant if $x \to y$ $\forall x \in X, y \notin X$

Claim: Let X, X' be two dominant sets, $X \neq X'$. Then either $X \subseteq X'$ or $X' \subseteq X$

Proof: Suppose not. then $\exists x \in X \setminus X'$, $x' \in X' \setminus X$. But then since both X, X' are dominant, $x \to x'$ & $x' \to x$. $\boxtimes$

Thus, dominant sets form a chain.
A top-cycle is the minimal element, or a smallest dominant set (must exist, since A is a dominant set)

Claim: For a tournament, the components can be numbered $C_1, \dots, C_k$ s.t. $\forall i < j$, every vtx. in C_i beats every vtx. in C_j
(easy)

Claim: The top-cycle $TC(T) = C_1$
(easy)

Claim: $TC(T)$ is Condorcet-consistent.
(easy)

Copeland: $sc(a) = |\{b : a \to b\}|$ = out-degree of a.
(or $\mu(a,b) > 0$)
Then $CO(T) = \{a :$ a has highest out-degree in $T\}$.

Claim: $CO(T) \subseteq TC(T)$
(easy)

Corollary: Copeland is Condorcet-consistent.

Given T, the dominion of a $D(a) = \{b : a \to b\}$
Say b "covers" a ($b \rhd a$) if $b \to a$ & $D(a) \subseteq D(b)$
a is uncovered if no alternative covers a
$UC(T)$ is the set of uncovered alternatives.

Claim: $UC(T) \neq \phi$

Proof: (1) if $a \rhd b$ & $b \rhd c$, then $a \rhd c$

(ii) if $a \rhd b$ then $b \ntriangleright a$

Thus, this gives a strict partial order on the alternatives. Given any $a \in A$, go backwards in this order, this gives an elt. of $UC(T)$.

Defn. : $a \in A$ is a king if $\forall b \neq a$, either $a \to b$ or $\exists c: a \to c, c \to b$.

Claim: $a \in UC(T)$ iff a is a king.

Proof: Fix $a \in UC(T)$. WTS: $\forall b \neq a$, either $a \to b$ or $\exists c: a \to c, c \to b$

Fix $b \neq a$, and suppose $a \not\to b$. Then $b \to a$.

Further either $\exists c: a \to c, c \to b$, or $\forall c: a \to c, b \to c$.

In the latter case, $b \rhd a$, & a is not uncovered.

Now suppose a is a king. WTS $a \in UC(T)$.

Suppose for a contradiction $\exists b: b \to a$ and $a \to c \Rightarrow b \to c$.

Then a does not have a path to b of length ≤ 2. & a is not a king ☒

Claim: $Co(T) \subseteq UC(T) \subsetneq TC(T)$

Proof: a has max out degree $\Rightarrow$ a is uncovered.

a is uncovered $\Rightarrow$ $a \in TC(T)$ (or $a \in C_1$)

(easy) ☒

So far we're only looked at the unwtd. tournament graph. Suppose we also consider wtd., randomized SCFs.

Defn: $\Delta(A) = \{x \in \mathbb{R}_+^m : \sum_{a \in A} x_a = 1\}$

$x \in \Delta(A)$ is a lottery over A (or a prob. distribution)

We know: may not exist $a \in A$ s.t. $\forall b$, $\mu(a,b) > 0$

Does there exist a distribution p. s.t. $\forall b$, $\mathbb{E}_{a \sim p}[\mu(a,b)] \geq 0$?

Define $M \in \mathbb{Z}^{m \times m}$: $M_{a,b} = \mu(a,b)$

(then diagonal entries are 0, & $M_{a,b} = -M_{b,a}$, so $M = -M^T$ & M is skew-symmetric)

Then for $p \in \Delta(A)$, $p^T M = \left(\sum_{a \in A} p_a \mu(a,b)\right)_{b \in A} = \left(\mathbb{E}_{a \sim p}[\mu(a,b)]\right)_{b \in B}$

& $p^T M q = \sum_{a,b \in A} p_a q_b \mu(a,b)$

Defn: A lottery p is maximal if $\forall q$, $p^T M q \geq 0$

$\Leftrightarrow \quad p^T M \geq 0$

Theorem: Maximal lotteries exist

Proof: Consider:

$$\begin{array}{lll} \max & \lambda & \\ M^T p & \geq \lambda & \times q \\ \sum_a p_a & = 1 & \times \mu \\ p & \geq 0 & \end{array} \qquad \qquad \begin{array}{ll} \min & \mu \\ -q^T M & \leq \mu \\ \sum_a q_a & = 1 \\ q & \geq 0 \end{array}$$

Both feasible so both have finite optimal solns. say (p^*, λ^*), (q^*, μ^*)

Then $\lambda^* = \mu^*$.

Suppose $\lambda^* > 0$. then $0 < q^{*T} M^T p^* = -q^{*T} M p^*$

$\leq -\mu^* p^*$

< 0 ☒

Theorem: Let p be any maximal lottery.

① If a^* is a Condorcet winner then $p_a = 1$.

(ii) If $\hat{a}$ is a Condorcet loser then $p_b = 0$

Proof: For ①: let $p = 1_{a=a^*}$. Then $\forall b$, $p^T M = \mu(a,b) > 0$ hence p is maximal.

Now consider any lottery p, and $q = 1_{a=a^*}$.

Then $p^T M q = \sum_a p_a \mu(a, a^*) = \sum_{a \neq a^*} p_a \mu(a, a^*)$

If $\exists a \neq a^*$ s.t. $p_a > 0$, then this is $-$ve.

For (ii)

Claim: If p is a maximal lottery, then $\forall b$: $p_b > 0$. $(p^T M)_b = 0$

Proof: Else $p^T M p > 0$, but $p^T M p = 0$ for skew-symm. matrices.

Proof of (ii) : $\forall p$, $(p^T M)_{\hat{a}} = \sum_a p_a \mu(a, \hat{a})$. Now $\mu(a, \hat{a}) > 0$ $\forall a \neq \hat{a}$

$= \sum_{a \neq \hat{a}} p_a \mu(a, \hat{a})$

if $\exists a \neq \hat{a}$ s.t. $p_a > 0$ then $(p^T M)_{\hat{a}} > 0$

so: $(p^T M)_{\hat{a}} = 0 \Rightarrow p_{\hat{a}} = 1$, but then p is not maximal

so for any maximal lottery, $(p^T M)_{\hat{a}} > 0$

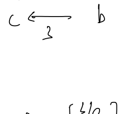
By the claim, this means $p_{\hat{a}} = 0$.

Example:

2: $a > b > c$

4: $b > c > a$

3: $c > a > b$



here : $M = \begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & -5 \\ b & -1 & 0 & 3 \\ c & 5 & -3 & 0 \end{array}$ $\quad p = \begin{bmatrix} 3/9 \\ 5/9 \\ 1/9 \end{bmatrix}$